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FINITE ELEMENT METHOD

Biswanath Banerjee & Amit Shaw
Department of Civil Engineering, IIT Kharagpur

Module 02: Weak Form & Rayleigh-Ritz method

Lecture 06 : Elements of Calculus of Variations - I

Concepts Covered

- **Introduction: Functional**
- **Euler-Lagrange Equation**
- **Problem**

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Functional ??

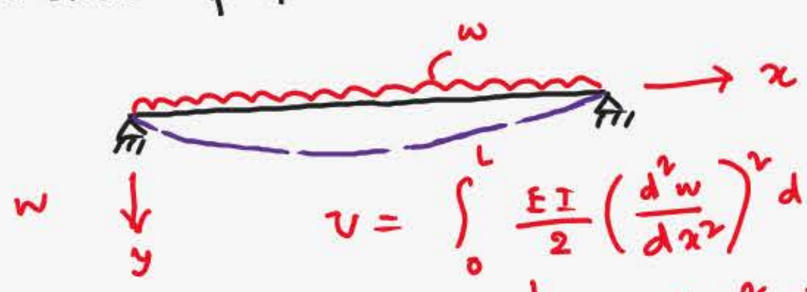
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Function of function

$f(x)$

$f'(x) \rightarrow$ Extremum

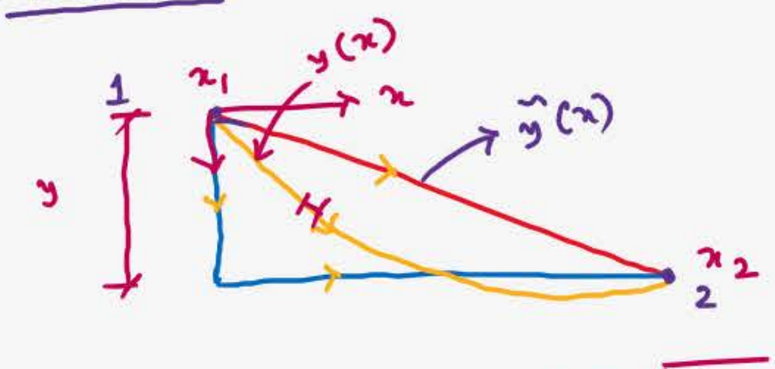
$f''(x)$



$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 dx \qquad U = \int \frac{M^2}{2EI}\,dx$$

$$\pi = \int_0^L \frac{EI}{2}\left(\frac{d^2w}{dx^2}\right) dx - \int_0^L q\,w\,dx \qquad M = \frac{d^2w}{dx^2}\cdot EI$$

Brachistochrone Problem :—



- Free fall
- under the action of gravity
- Frictionless.
- which path will take min^{m} time.

$$v = \sqrt{2gy}$$

$$\underline{I} = \int_{x_1}^{x_2} \frac{ds}{v} \qquad ds = (dx^2 + dy^2)^{1/2}$$

$$= \int_{x_1}^{x_2} \frac{1}{\sqrt{2gy}}\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} dx = \int_{x_1}^{x_2} \frac{(1+y'^2)^{1/2}}{(2gy)^{1/2}}\,dx$$

$$I = \int_{x_1}^{x_2} F(x, y, y')\,dx$$

$y(x) \rightarrow$ optimal path

$\tilde{y}(x) \rightarrow$ Approximate path.

$$\rightarrow \underline{\tilde{y}(x) = y(x) + \epsilon\eta(x)}$$

$\epsilon = 0$

$y(x_1) = y_1$

$y(x_2) = y_2$

$\begin{cases} \eta(x_1) = 0 \\ \eta(x_2) = 0 \end{cases}$

$$\tilde{I} = \int_{x_1}^{x_2} \underline{F(x, \tilde{y}, \tilde{y}')}\,dx$$

$$= \int_{x_1}^{x_2} F(x,\ y + \epsilon\eta,\ y' + \epsilon\eta')\,dx$$

$$= I(\epsilon)$$

$$\boxed{\left.\frac{d\tilde{I}}{d\epsilon}\right|_{\epsilon=0} = 0}$$

$$\tilde{I} = \tilde{I}\Big|_{\epsilon=0} + \left.\frac{d\tilde{I}}{d\epsilon}\right|_{\epsilon=0}\epsilon + \left.\frac{d^2\tilde{I}}{d\epsilon^2}\right|_{\epsilon=0}\frac{\epsilon^2}{L^2} + \cdots\cdots$$

$$\tilde{I} - I = \left.\frac{d\tilde{I}}{d\epsilon}\right|_{\epsilon=0} + \left.\frac{d^2\tilde{I}}{d\epsilon^2}\right|_{\epsilon=0}\frac{\epsilon^2}{L^2}$$

$$\tilde{I} = \int_{x_1}^{x_2} F(x,\ y + \epsilon\eta,\ y' + \epsilon\eta')\,dx \qquad \tilde{y} = y + \epsilon\eta$$

$$\frac{d\tilde{I}}{d\epsilon} = \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial \tilde{y}}\cdot\frac{d\tilde{y}}{d\epsilon} + \frac{\partial F}{\partial \tilde{y}'}\cdot\frac{d\tilde{y}'}{d\epsilon}\right)dx$$

$$= \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial \tilde{y}}\eta + \frac{\partial F}{\partial \tilde{y}'}\cdot\eta'\right)dx$$

$$\left.\frac{d\tilde{I}}{\partial\epsilon}\right|_{\epsilon=0} = \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right)dx$$

$$= \int_{x_1}^{x_2}\frac{\partial F}{\partial y}\eta\,dx + \left.\frac{\partial F}{\partial y'}\eta\right|_{x=x_1}^{x=x_2} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta\,dx$$

$$= \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)\,dx + \left.\frac{\partial F}{\partial y'}\eta\right|_{x=x_1}^{x=x_2}$$

($\eta = 0$ at both x_1 and x_2, so the boundary term vanishes)

$$\left.\frac{d\tilde{I}}{d\epsilon}\right|_{\epsilon=0} = 0 \Rightarrow \boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0} \quad \text{for any } \eta(x)$$

Euler - Lagrange Equation

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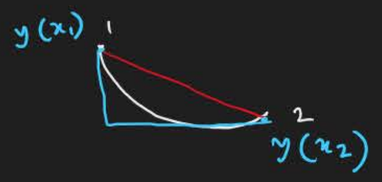
Module 02: Weak Form & Rayleigh-Ritz method

Lecture 07 : Elements of Calculus of Variations - II

Concepts Covered

- **Euler-Lagrange Equation and Boundary Condition**
- **Problem**

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$$I = \int_{x_1}^{x_2} \left(\frac{1+y'^2}{2gy}\right)^{1/2} dx \qquad y' = \frac{dy}{dx}$$

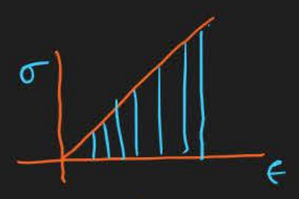
$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx \qquad \tilde{y}(x) = y(x) + \epsilon\eta(x)$$

$$\eta(x_1) = \eta(x_2) = 0$$

$$\left.\frac{d\tilde{I}}{d\epsilon}\right|_{\epsilon=0} = 0$$

$$\Downarrow$$

$$\text{EL Equation} - \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \forall\, \eta(x)$$



$\frac{1}{2}\sigma\epsilon$

Bar



$$U = \int_0^L \frac{AE}{2}\left(\frac{du}{dx}\right)^2 dx$$

$$\Pi = \int_0^L \frac{AE}{2}\left(\frac{du}{dx}\right)^2 dx - \int_0^L q(x)\,u(x)\,dx$$

$$= \int_0^L \left[\frac{AE}{2}\left(\frac{du}{dx}\right)^2 - qu\right] dx$$

$$= \int_0^L \left[\frac{AE}{2}u'^2 - qu\right] dx$$

$U = \int_0^L \frac{1}{2} E\epsilon^2 \frac{A}{2} E\left(\frac{du}{dx}\right)^2 dx$

$\epsilon = \frac{du}{dx}$

at $x = 0$, $u = 0$ → Dirichlet's BC / Kinematic BC

$AE\frac{du}{dx} = 0$

$\sigma \cdot \hat{n} = 0$

$AE\frac{du}{dx} = ?$ → Neumann's BC / Force BC

$$\frac{\partial F}{\partial u} = -q(x), \qquad \frac{\partial F}{\partial u'} = AEu' = AE\frac{du}{dx}$$

EL Equn → $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

EL Equn for the Bar → $\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0$

$$\Rightarrow -q(x) - \frac{d}{dx}\left(AE\frac{du}{dx}\right) = 0$$

$$\Rightarrow \boxed{AE\frac{d^2u}{dx^2} + q = 0}$$

$\frac{\partial F}{\partial y'}\eta(x)\Big|_{x=x_1}^{x=x_2}$

$\eta(x_1) = \eta(x_2) = 0$ for Brachistochrone Problem.

$x = 0$, $u = 0$ — $x = L$, $u = ??$, $\sigma \cdot \hat{n} = 0$, $AE\frac{du}{dx} = 0$

$$AE\frac{d^2u}{dx^2} \quad q = 0 \qquad 0 < x < L$$

$u = 0$ at $x = 0$

$AE\frac{du}{dx} = 0$ at $x = L$

Delta operator:

$\delta y = \tilde{y} - y(x)$

↓

variation of y

$F(x, y, y')$

$$\rightarrow \delta F = \frac{\partial F}{\partial y}\cdot dy + \frac{\partial F}{\partial y'}dy'$$

$$dF = \frac{\partial F}{\partial x}\cdot dx + \frac{\partial F}{\partial y}\cdot dy + \frac{\partial F}{\partial y'}dy'$$

$\delta y = \tilde{y} - y$

$\delta y' = \delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)$

$\int \delta y\, dx = \delta \int y\, dx$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\tilde{I} = \int_{x_1}^{x_2} F(x, y+\delta y, y'+\delta y')\, dx$$

$$\tilde{I} - I = \delta^{(1)} I = 0$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\cdot\delta y + \frac{\partial F}{\partial y'}\delta y'\right] dx$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \forall\, \delta y$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}(\delta y)'\right] dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\, dx + \left.\frac{\partial F}{\partial y'}\cdot\delta y\right|_{x=x_1}^{x=x_2}$$

BC: Either $\frac{\partial F}{\partial y'}$ or δy is given at $x = x_1$

or $\frac{\partial F}{\partial y'}$ or δy is given at $x = x_2$

$$I = \int_{x_1}^{x_2} F(x, y, y', y'')\, dx$$

$$\tilde{I} = \int_{x_1}^{x_2} F(x, y+\delta y, y'+\delta y', y''+\delta y'')\, dx$$

$$\Pi = \int_0^L \frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 dx - \int_0^L q\,w\, d$$



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Module 02: Weak Form & Rayleigh-Ritz method

Lecture 08 : Elements of Calculus of Variations - III

Concepts Covered

- First variation of a functional involving higher order derivatives.
- Example: Euler-Bernoulli Beam.
- First variation of a functional involving several dependent variables.

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/ Functional, $I = \int_{x_1}^{x_2} F(x, y, y')\, dx$

/ First variation of I, $\delta^{(1)} I = 0$

/ Euler-Lagrange Eqn, $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

BC $\begin{cases} \text{at } x = x_1 & \text{either } \left.\frac{\partial F}{\partial y'}\right|_{x=x_1} \text{ or } y(x_1) \text{ given} \\ \text{at } x = x_2 & \text{either } \left.\frac{\partial F}{\partial y'}\right|_{x=x_2} \text{ or } y(x_2) \text{ given} \end{cases}$

Kinematic or Dirichlet

Force or Neumann

Functionals involving higher order derivative :- First variation.

$$I = \int_{x_1}^{x_2} F(x, y, y', y'')\, dx$$

$$\delta^{(1)} I = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\cdot \delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y''}\delta y''\right] dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\, dx + \left.\frac{\partial F}{\partial y'}\delta y\right|_{x=x_1}^{x=x_2} + \left.\frac{\partial F}{\partial y''}\cdot\delta y'\right|_{x=x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\delta y'\, dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\, dx + \left.\frac{\partial F}{\partial y'}\delta y\right|_{x=x_1}^{x=x_2} + \left.\frac{\partial F}{\partial y''}\delta y'\right|_{x=x_1}^{x=x_2} - \left.\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\delta y\right|_{x=x_1}^{x=x_2} + \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)\delta y\, dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)\right]\delta y\, dx + \left[\frac{\partial F}{\partial y'} - \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\right]\delta y\Big|_{x=x_1}^{x=x_2} + \left.\frac{\partial F}{\partial y''}\cdot\delta y'\right|_{x=x_1}^{x=x_2}$$

EL Eqn $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$

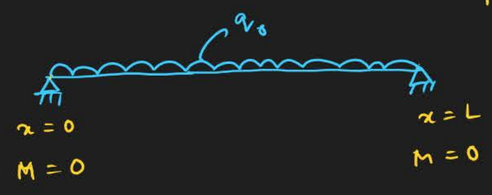
at $x = x_1$ either $\left[\frac{\partial F}{\partial y'} - \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\right]$ or $y(x_1)$ given

at $x = x_2$ either $\left[\frac{\partial F}{\partial y'} - \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\right]$ or $y(x_2)$ given

at $x = x_1$ either $\left.\frac{\partial F}{\partial y''}\right|_{x=x_1}$ or $\left.\frac{dy}{dx}\right|_{x=x_1}$ is given.

at $x = x_2$ either $\left.\frac{\partial F}{\partial y''}\right|_{x=x_2}$ or $\left.\frac{dy}{dx}\right|_{x=x_2}$ is given.

Example:



$$\Pi = \int_0^L \frac{M^2}{2EI}\, dx - \int_0^L q_0 w\, dx, \quad M = -EI\frac{d^2w}{dx^2}$$

$$\Pi = \int_0^L \frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 dx - \int_0^L q_0 w\, dx$$

$$= \int_0^L \underbrace{\left[\frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 - q_0 w\right]}_{F(x, w, w'')} dx$$

EL Eqn $\frac{\partial F}{\partial w} - \frac{d}{dx}\left(\frac{\partial F}{\partial w'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial w''}\right) = 0$

$\frac{\partial F}{\partial w'} = 0$

$$-q_0 + \frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = 0 \Rightarrow \boxed{\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = q_0}, \quad 0 < x < L$$

$$\Rightarrow \boxed{EI\frac{d^4w}{dx^4} = q_0}$$

① $\begin{cases} \frac{\partial F}{\partial w'} - \frac{d}{dx}\left[\frac{\partial F}{\partial w''}\right] \text{ or } w \\ \text{at } x = 0 \quad w = 0 \\ \text{at } x = L \quad w = 0 \end{cases}$

$$-\frac{d}{dx}\left(\frac{\partial F}{\partial w''}\right) = -\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = -EI\frac{d^3w}{dx^3}$$

② $\begin{cases} \frac{\partial F}{\partial w''} \text{ or } w' = \frac{dw}{dx} \\ \text{at } x = 0 \quad EI\frac{d^2w}{dx^2} = 0 \\ \text{at } x = L \quad EI\frac{d^2w}{dx^2} = 0 \end{cases}$

$$\frac{\partial F}{\partial w''} = EI\frac{d^2w}{dx^2}$$

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx, \qquad I = \int_{x_1}^{x_2} F(x, y, y', y'')\, dx$$

$$I = \int_{x_1}^{x_2} F(x, y_1, y_1', y_2, y_2')\, dx$$

$$\delta^{(1)}_{y_1} I = 0, \qquad \delta^{(1)}_{y_2} I = 0$$

EL Eqn $\frac{\partial F}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1'}\right) = 0$

$\frac{\partial F}{\partial y_2} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2'}\right) = 0$

BC either $\frac{\partial F}{\partial y_1'}$ or y_1 at $x = x_1, x_2$

either $\frac{\partial F}{\partial y_2'}$ or y_2 at $x = x_1, x_2$

$$I = \int_{x_1}^{x_2} F(x, y_1, y_1', y_1'', y_2, y_2', y_2'')\, dx$$



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Module 02: Weak Form & Rayleigh-Ritz method

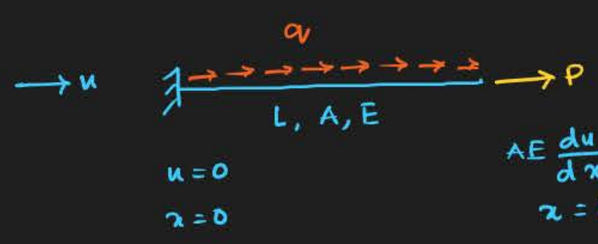
Lecture 09 : Strong Form and Weak Form

Concepts Covered

- Euler – Lagrange Equation: Strong Form
- Examples – Bar and Beam.
- Weak Form
- Bilinear and Linear Form

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$$I = \int_{x_1}^{x_2} F(x, y, y', y'')\, dx \quad , \quad \underline{\delta^{(1)} I = 0}$$



$$\Pi = \int_0^L \frac{AE}{2}\left(\frac{du}{dx}\right)^2 dx - \int_0^L qu\, dx - P.u(L) \quad \text{— PE}$$

$\delta^{(1)} I = 0$ → Strong Form

$$AE\frac{d^2u}{dx^2} = q \qquad 0 < x < L$$
$$u = 0 \quad \text{at} \quad x = 0$$
$$AE\frac{du}{dx} = P \quad \text{at} \quad x = L$$
} DE

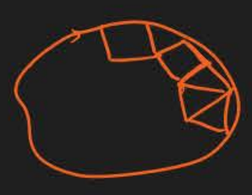


$$\Pi = \int_0^L \frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 dx - \int_0^L qw\, dx \quad \text{— PE}$$

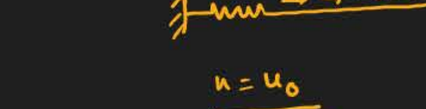
$\delta^{(1)} I = 0$ → Strong Form

$$EI\frac{d^4w}{dx^4} = q_0$$
$$w(0) = w(L) = 0$$
$$EI\frac{d^2w}{dx^2}\Big|_{x=0} = EI\frac{d^2w}{dx^2}\Big|_{x=L} = 0$$
} DE

$\rightarrow w(x) \cong C_0 + C_1 x + C_2 x^2 + C_3 x^3 \cdots$



Weak form:



$$AE\frac{d^2u}{dx^2} = q_0 \qquad 0 < x < L$$
$$u = 0 \qquad x = 0$$
$$AE\frac{du}{dx} = P \qquad x = L$$

Weight Function v

$$\int_0^L v\left(AE\frac{d^2u}{dx^2} - q_0\right) dx = 0$$

$$\Rightarrow \underbrace{AE\frac{du}{dx}v\Big|_{x=0}^{x=L}}_{} - \int_0^L \frac{dv}{dx} AE\frac{du}{dx}\, dx - \int_0^L q_0 v\, dx = 0$$

Admissibility Criteria. $v\big|_{x=0}^{x=L} = 0$

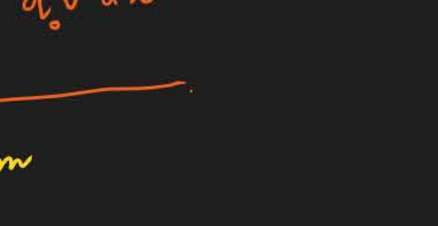
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Homogeneous BC of v



Natural BC appears automatically

Weighted Integral statement of the strong form.



Bilinear Form & Linear Form:

$$f(x) = x^2 \rightarrow f(x+y) = (x+y)^2 = x^2 + 2xy + y^2 = f(x) + f(y) + 2xy$$

Def. $f(ax + by) = af(x) + bf(y) \qquad a, b \in \mathbb{R}$

$l(v) \qquad l(av + bw) = a\,l(v) + b\,l(w) \qquad a, b \in \mathbb{R}$ ← Linear Form

$B(v, u)$ weak statement of the Bar Problem.

$$\underbrace{AE\frac{du}{dx}v\Big|_{x=0}^{x=L}}_{} - \int_0^L AE\frac{dv}{dx}\cdot\frac{du}{dx}\cdot dx - \int_0^L q_0 v\, dx = 0$$

$B(v,u)$ ← Linear form

$$l(v) = -\int_0^L q_0 v\, dx + AE\frac{du}{dx}\Big|_{x=L}$$

$$B(v, u) = \int_0^L \frac{dv}{dx} AE\frac{du}{dx}\, dx = B(u, v)$$

Cond. BF $\quad B(\alpha_1 v_1 + \alpha_2 v_2, \beta_1 u_1 + \beta_2 u_2) = \alpha_1\beta_1 B(v_1, u_1) + \alpha_2\beta_1 B(v_2, u_1) + \alpha_1\beta_2 B(v_1, u_2) + \alpha_2\beta_2 B(v_2, u_2)$

$\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$

$B(v, u) = B(u, v)$ ← Symm. Bilinear form.

$\boxed{B(v, u) = l(v)}$ weak statement

Bar

$$B(v,u) = \int_0^L \frac{dv}{dx} AE\frac{du}{dx}\, dx.$$
$$l(v) = -\int_0^L q_0 v\, dx + PV(L)$$

$\Pi \xrightarrow{\delta^{(1)}\Pi = 0}$ EL (strong form) $\xrightarrow{\int EL = 0}$ weak form.



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Module 02: Weak Form & Rayleigh-Ritz method

Lecture 10 : Rayleigh-Ritz method - I

Concepts Covered

- **Bilinear and Linear Form and Quadratic Functional**
- **Rayleigh-Ritz method**

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Recap:

- Strong form of a Potential function. —— EL Eqⁿ
- Variational statement or weak statement
- Integration by parts.
  - (a) reduces diff. requirement
  - (b) Natural BC satisfied.



$$\Pi = \int_0^L \frac{AE}{2}\left(\frac{du}{dx}\right)^2 dx - \left[\int_0^L q_0 u\,dx - u(L)\cdot P_0\right]$$

- $B(v,u) = \ell(v)$ — u, v
- $B(v,u) = \int_0^L \frac{dv}{dx} AE \frac{du}{dx}\,dx$, $\ell(v) = \int_0^L q_0 v\,dx + v(L)P_0$
- $B(v,u) = B(u,v)$
- we can derive quadratic functional.

$$B(v,u) = \int_0^L \frac{dv}{dx} AE \frac{du}{dx}\,dx$$

$$B(\delta u, u) = \int_0^L \frac{d\delta u}{dx} AE \frac{du}{dx}\,dx \quad \underline{v = \delta u} \quad = \int_0^L \delta\left(\frac{du}{dx}\right) AE \frac{du}{dx}\,dx$$

$$= \delta \int_0^L \frac{AE}{2}\left(\frac{du}{dx}\right)^2 dx$$

$$= \frac{1}{2}\delta\left[B(u,u)\right]$$

$$\ell(\delta u) = \int_0^L q_0 \delta u\,dx + \delta u(L) P_0 \quad = \quad \delta \ell(u)$$

$$B(\delta u, u) = \ell(\delta u) \Rightarrow \quad B(\delta u, u) - \ell(\delta u) = 0$$

$$\Rightarrow \frac{1}{2}\delta[B(u,u)] - \delta[\ell(u)] = 0$$

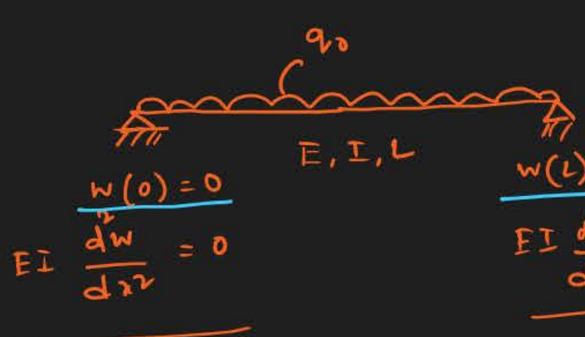
$$\Rightarrow \delta\left[\frac{1}{2}B(u,u) - \ell(u)\right] = 0$$

$$\Rightarrow \boxed{\delta^{(1)}\Pi = 0}$$

$B(\delta u, u) = B(u, \delta u)$

$\Pi \xrightarrow{\delta^{(1)}\Pi = 0}$ strong form $\xrightarrow[\text{Integral statement}]{\text{weighted}}$ weak form

Symm. Bilinear form



$$\Pi = \int_0^L \frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 dx - \int_0^L q_0 w\,dx$$

$$\begin{cases} EI\frac{d^4w}{dx^4} = q_0 , & w(0) = w\,EI\frac{d^2w}{dx^2}\Big|_{x=0} = 0 \\ & w(L) = EI\frac{d^2w}{dx^2}\Big|_{x=L} = 0 \end{cases}$$

$$B(v,w) = \int_0^L \frac{d^2v}{dx^2} EI \frac{d^2w}{dx^2}\,dx \quad \left\{ v \text{ weight function} \right.$$

$$\ell(v) = \int_0^L q_0 v\,dx$$

$$B(v,w) = \ell(v)$$

Rayleigh - Ritz method:—

$$u(x) = \sum_{j=1}^{n} c_j \Phi_j(x) + \Phi_0(x) \qquad u(x) = c_1\Phi_1 + c_2\Phi_2 + \cdots$$

(c_j: Ritz's coefficient)

$B(v,u) = \ell(v)$, $v = \Phi_i$ for $i = 1$ to n

$$B\left(\Phi_i, \sum_{j=1}^{n} c_j \Phi_j + \Phi_0\right) = \ell(\Phi_i)$$

$$\Rightarrow B\left(\Phi_i, \sum_{j=1}^{n} c_j \Phi_j\right) = \ell(\Phi_i) - B(\Phi_i, \Phi_0)$$

$$\Rightarrow \sum_{j=1}^{n} B(\Phi_i, \Phi_j) c_j = \ell(\Phi_i) - B(\Phi_i, \Phi_0)$$

for $i = 1, \ldots, n$

Given $u = u_0 \rightarrow$ $\quad u(x_0) = \sum_{j=1}^{n} c_j \Phi_j(x_0) + \Phi_0(x_0)$

$= 0 + u_0$

$\leftarrow [B]\{c\} = \{f\}$ ($n \times n$, $n \times 1$, $n \times 1$)

Bar:

$$B_{ij} = \int_0^L \frac{d\Phi_i}{dx} AE \frac{d\Phi_j}{dx}\,dx.$$

- Linearly independent col^{n} ⟶ Full rank matrix
- Complete polynomial. $\quad 1 \quad x \quad x^2$
- $\Phi_i(x_0) = 0$
-



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THANK YOU !





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FINITE ELEMENT METHOD

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Module 02: Weak Form & Rayleigh-Ritz method

Lecture 11 : Rayleigh-Ritz method - II

Concepts Covered

- **Rayleigh-Ritz method**
- **Problem**

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Recap:

$$\sum_{j=1}^{n} B(\phi_i, \phi_j) c_j = \ell(\phi_i) - B(\phi_i, \phi_0) \quad \text{for } i = 1, (1) n$$

$$u = \sum_{j=1}^{n} c_j \phi_j(x) + \phi_0$$

Problem

$$-\frac{d}{dx}\left[(1+x)\frac{du}{dx}\right] = 0 \quad \text{for } 0 < x < 1$$

$$u(0) = 0$$
$$u(1) = 1$$

— weak form, w

$$\int_0^1 w\left[-\frac{d}{dx}\left((1+x)\frac{du}{dx}\right)\right] dx = 0$$

$$\Rightarrow \int_0^1 \frac{dw}{dx}(1+x)\frac{du}{dx}\,dx - \underbrace{w(1+x)\frac{du}{dx}\Big|_{x=0}^{x=1}}_{=0} = 0$$

— R-R method:

$$u(x) = \sum_{j=1}^{n} c_j \phi_j + \phi_0$$

weight function $w = \phi_i$

$$\sum_{j=1}^{n}\left[\int_0^1 \frac{d\phi_i}{dx}(1+x)\frac{d\phi_j}{dx}\,dx\right] c_j + \int_0^1 \frac{d\phi_i}{dx}(1+x)\frac{d\phi_0}{dx}\,dx = 0$$

$$\sum_{j=1}^{n} B(\phi_i, \phi_j) c_j = \ell(\phi_i) \qquad i = 1(1)n$$

BC

$u(0) = 0$, $u(1) = 1$

$\phi_i(0) = 0$, $\phi_i(1) = 0$

$\phi_0(0) = 0$, $\phi_0(1) = 1$ $\Rightarrow \phi_0 = x$

$\phi_1 = x(1-x)$, $\phi_2 = x^2(1-x)$

$\{\phi_1 = x^2(1-x) = x^2 - x^3$, $\phi_2 = x^3(1-x) = x^3 - x^4\}$ ✗

$n = 2$.

$$u(x) = c_1\phi_1 + c_2\phi_2 + \phi_0$$
$$= c_1 x(1-x) + c_2 x^2(1-x) + x$$

$B_{ij}\ c_i = f_i$

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

$$B_{11} = \int_0^1 \frac{d\phi_1}{dx}(1+x)\frac{d\phi_1}{dx}\,dx$$

$$\ell_1 = \int_0^L \frac{d\phi_1}{dx}(1+x)\frac{d\phi_0}{dx}\,dx = -f_1$$

$$\frac{d\phi_1}{dx} = 1 - x - x = 1 - 2x$$

$$f_1 = -\int_0^L (1-2x)(1+x)\,1\,dx$$

$$B_{11} = \int_0^1 (1+x)(1-2x)^2\,dx$$

$$= \frac{1}{6}$$

$$= \frac{1}{2}$$

$$B_{12} = \int_0^1 \frac{d\phi_1}{dx}(1+x)\frac{d\phi_2}{dx}\,dx = \frac{17}{60}, \quad B_{21} = B_{12} = \int_0^1 \frac{d\phi_2}{dx}(1+x)\frac{d\phi_1}{dx}\,dx = \frac{17}{60}$$

$$\frac{d\phi_2}{dx}$$

$$B_{22} = \int_0^1 \frac{d\phi_2}{dx}(1+x)\frac{d\phi_2}{dx}\,dx$$

$$f_2 = -\int_0^1 (1+x)\frac{d\phi_2}{dx}\cdot\frac{d\phi_0}{dx}\,dx$$

$$= -\int_0^1 (1+x)\frac{d\phi_2}{dx}\cdot 1\,dx$$

$$= \frac{7}{30}$$

$$= \frac{1}{12}$$

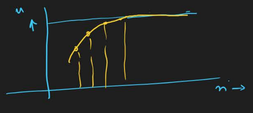
$$[B]\{C\} = \{f\}$$

$$\begin{bmatrix} 1/2 & 17/60 \\ 17/60 & 7/30 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 1/6 \\ 1/12 \end{Bmatrix}$$

$c_1 = \frac{55}{131}$, $c_2 = -\frac{20}{131}$

$$u(x) = \frac{55}{131}x(1-x) - \frac{20}{131}x^2(1-x) + x \qquad n = 2$$

$$u_{exact} = \frac{\log(1+x)}{\log 2}$$



$n = 3$. $u_{approx} \approx u_{exact}$



$$w(0) = 0, \quad \frac{dw}{dx} = w'(L) = 0$$

$$EI\frac{d^2w}{dx^2} = M_0$$

$$\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = 0$$

$$\Rightarrow EI\frac{d^3w}{dx^3} = 0$$

$$w(x) = c_1\phi_1 + c_2\phi_2 + \phi_0 \qquad n = 2$$

$$\phi_0 = 0$$

$$\phi_i(0) = 0, \quad \phi_i'(0) = 0 = \left.\frac{d\phi_i}{dx}\right|_{x=0}$$

$$\phi_1 = x^2$$
$$\phi_2 = x^3.$$

$n = 2$

$$w(x) = c_1 x^2 + c_2 x^3.$$

$$B_{ij} = \int_0^L \frac{d^2\phi_i}{dx^2}\,EI\,\frac{d^2\phi_j}{dx^2}\,dx$$

$$\ell(\phi_i) = \ell_i = \int_0^L q_0\phi_i\,dx + \phi_i(L)M_0$$

$$= \int_0^L q_0\phi_i\,dx + \left.\frac{d\phi_i}{dx}\right|_{x=L} M_0$$

$$B_{11} = \int_0^L \frac{d^2\phi_1}{dx^2}\,EI\,\frac{d^2\phi_1}{dx^2}\,dx$$

$$\ell_1 = \int_0^L q_0\phi_1\,dx + \left.\frac{d\phi_1}{dx}\right|_{x=L} M_0$$

$\phi_1 = x^2$

$\frac{d\phi_1}{dx} = 2x$

$\frac{d^2\phi_1}{dx^2} = 2$

$\phi_2 = x^3$

$$= \int_0^L 2EI.2\,dx$$

$$= 4EIL$$

$$B_{12} = 6L^2EI$$

$$b_{22} = 12L^3EI$$

$$\ell_1 = \int_0^L q_0 x^2\,dx + 2LM_0$$

$$= \frac{q_0L^3}{3} + 2M_0L$$

$$\ell_2 = \frac{1}{4}q_0L^4 + 3M_0L^2$$

$$EI\begin{bmatrix} 4L & 6L^2 \\ 6L^2 & 12L^3 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \frac{q_0L^3}{12}\begin{Bmatrix} 4 \\ 3L \end{Bmatrix} + M_0L\begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$$

$$c_1 = \frac{5q_0L^2 + 12M_0}{24EI}, \qquad c_L = -\frac{q_0L}{12EI}$$

$$w_{approx.} = c_1\phi_1 + c_2\phi_2$$
$$= \frac{5q_0L^2 + 12M_0}{24EI}x^2 - \frac{q_0L}{12EI}x^3$$

$n = 3$.

$\phi_1 = x^2$, $\phi_2 = x^3$, $\phi_3 = x^4$ $\qquad w(x) = c_1\phi_1 + c_2\phi_2 + c_3\phi_3$

w_{exact}

$$EI\begin{bmatrix} 4 & 6L & 8L^2 \\ 6L & 12L^2 & 18L^3 \\ 8L^2 & 18L^3 & \frac{144}{5}L^4 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}q_0L^2 + 2M_0 \\ \frac{1}{4}q_0L^3 + 3M_0L \\ \frac{1}{5}q_0L^4 + 4M_0L^2 \end{Bmatrix}$$

c_1, c_2, c_3

$n = 4$ $\qquad w = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_4\phi_4$, $\phi_1 = x^2$, $\phi_2 = x^3$, $\phi_3 = x^4$, $\phi_4 = x^5$

$$\Pi = \int_0^L \frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 dx - \int_0^L q_0 w\,dx - \left.\frac{dw}{dx}\right|_{x=L} M_0$$

$$w(x) = c_1\phi_1 + c_2\phi_2 + \phi_0$$

$$\frac{d\Pi}{dc_1} = 0 \qquad \frac{d\Pi}{dc_2} = 0$$



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